Online Appendix A: Proofs

Proof of Proposition 1. By plugging Equation (12) into (11), it is easy to see that any steady state with positive capital stock $K^* > 0$ and such that $\theta^* < 1$ is identified by the equation:

$$K^* = \frac{(1-\varphi)(1+\alpha A (K^*)^{\alpha-1} - \gamma)}{\sigma[1 + A(K^*)^{\alpha-1}]^2} \cdot \left(\Gamma - \frac{\Delta}{4}\right) \cdot (1-\alpha) A (K^*)^{\alpha},$$

which can be rewritten as:

$$c \cdot [(K^*)^{1-\alpha} + A]^2 = \left[(1-\gamma)(K^*)^{(1-\alpha)} + \alpha A\right] \cdot A, \tag{A1}$$

where $c \equiv \frac{\sigma}{(1-\varphi)\cdot\left(\Gamma - \frac{\Delta}{4}\right)\cdot(1-\alpha)}$. We can verify the above equation admits a unique solution $K^* > 0$ provided $c < \alpha$, which imposes an upper bound on σ.

Before studying the steady state, we need to verify that $\theta^* < 1$ (for all investors). From Equation (12), the household closest to a manager invests a share of wealth:

$$\theta(K_t^{1-\alpha}) = \frac{K_t^{1-\alpha}[(1-\gamma)K_t^{1-\alpha} + \alpha A]}{c \cdot [K_t^{1-\alpha} + A]^2} \cdot z,$$

where $z = \frac{\Gamma}{\left(\Gamma - \frac{\Delta}{4}\right)\cdot(1-\alpha)}$. The function $\theta(\cdot)$ is increasing in $K_t^{1-\alpha}$ provided $K_t^{1-\alpha} < A$, which as we show is strictly satisfied at the steady state capital level, and thus along transitional dynamics around the steady state. This implies that starting from a below steady state level of capital stock, risk taking increases over time until the steady state is reached. As a consequence, by exploiting Equation $(A1)$, all investors set an interior level of risk taking at the steady state provided $(K^*)^{1-\alpha} < A/z$, where $z > 1$. By plugging this condition into $(A1)$, we find this is equivalent to:

$$c > \frac{z \cdot [(1-\gamma) + \alpha z]}{(1+z)^2},$$

which imposes a lower bound on σ. The upper and lower bounds are mutually compatible, namely $\frac{z\cdot[(1-\gamma)+\alpha z]}{(1+z)^2} < \alpha$, provided $2\alpha > (1-\gamma)$, which we assume to hold. This analysis thus identifies variance bounds $\bar{\sigma}$ and $\underline{\sigma}$, with $\bar{\sigma} > \underline{\sigma}$, to which we restrict the analysis of our model.

Consider the steady state prevailing for $\sigma \in \left(\underline{\sigma}, \bar{\sigma}\right)$. This is identified by Equation $(A1)$. By applying the implicit function theorem, and after some algebra, one can find that:

$$\frac{d(K^*)^{1-\alpha}}{dA} \propto -\frac{-c(K^*)^{2(1-\alpha)} + cA^2 - \alpha A^2}{2c[(K^*)^{1-\alpha} + A] - (1-\gamma)A} > 0, \tag{A2}$$

$$\frac{d(K^*)^{1-\alpha}}{dc} \propto -\frac{[(K^*)^{1-\alpha} + A]^2}{2c[(K^*)^{1-\alpha} + A] - (1-\gamma)A} < 0, \tag{A3}$$

where both inequalities rely on the restriction $(K^*)^{1-\alpha} < A/z$ and $c < \alpha$. Condition (A2) intuitively says that the steady state capital stock increases in productivity A. Condition (A3) says that the steady state capital stock increases in the number of managers (because lower Δ reduces c).

Consider now the dynamics of the model. By exploiting Equations (11) and (12), one can write the law of motion for our model economy as:

$$K_t^{\alpha}\frac{(K_t^{1-\alpha}+A)^2}{[(1-\gamma)K_t^{1-\alpha}+\alpha A]}-\frac{1}{c}AK_{t-1}^{\alpha}=0. \qquad (A4)$$

The above difference equation implicitly defines a function $K_t(K_{t-1})$ whose slope is equal to:

$$\frac{dK_t}{dK_{t-1}}=\frac{\frac{1}{c}\cdot\alpha A}{K_{t-1}^{1-\alpha}\cdot\frac{(K_t^{1-\alpha}+A)^2}{[(1-\gamma)K_t^{1-\alpha}+\alpha A]}\left\{\frac{\alpha}{K_t^{1-\alpha}}+\left(\frac{1-\alpha}{K_t^{1-\alpha}+A}\right)\frac{[(1-\gamma)K_t^{1-\alpha}+(\alpha-1+\gamma)A]}{[(1-\gamma)K_t^{1-\alpha}+\alpha A]}\right\}}.$$

At the $K_t=K_{t-1}=0$ steady state, the above slope becomes equal to:

$$\frac{dK_t}{dK_{t-1}}=\frac{\alpha}{c}>1,$$

Where the inequality is due to the assumption $c<\alpha$. Thus, the zero capital steady state is unstable, and the mapping $K_t(K_{t-1})$ must cut the 45 degrees line at the interior steady state K^* with a slope less than one, implying that K^* is locally stable. The comparative statics of steady state capital K^* and risk taking θ^*follow by inspection from Equation (A1) and Equation (12) in the text. In fact, higher number of money managers reduces the previously defined parameter c, increasing K^* (by A3) and θ^* (by (A3)+(12)). On the other hand, higher productivity A increases K^* (by A2) by leaves θ^* unaffected (because θ^* in (12) depends on $AK^{\alpha-1}$ which stays constant in the long run).

Proof of Corollary 2 At the steady state capital sock $K^*(\Gamma,A)$, the new productivity level A' sets the wage rate, fees and intermediation at time t. In particular, investment and intermediation are pinned down by the equations:

$$K_{t+1}=\theta_{t+1}A(\frac{w_t}{A}),$$

$$\theta_{t+1}(K_{t+1},A)=\frac{(1-\varphi)(1+\alpha AK_{t+1}^{\alpha-1}-\gamma)}{\sigma[1+AK_{t+1}^{\alpha-1}]^2}\cdot\left(\Gamma-\frac{\Delta}{4}\right).$$

where $(\frac{w_t}{A})$ is by definition invariant to changes in A, for the initial capital stock is predetermined.

Consider the effects of a change in A. The impact of such change on investment and intermediation is determined by the behavior of the ratio $K_{t+1}/\theta_{t+1}(K_{t+1},A)A$. By the proof of proposition 1 we know that this ratio is an increasing function of K_{t+1} and a decreasing function of A at the steady state capital level. As a result, by the implicit function theorem, a drop in productivity reduces financial intermediation and the capital stock K_{t+1}. The relative size of the financial sector depends on the effect of the productivity change on the product $AK_{t+1}^{\alpha-1}$. Denote $x\equiv K_{t+1}^{1-\alpha}/A$. The relative size of finance increases with x. In this regard, note that the equilibrium condition $\frac{K_{t+1}}{\theta_{t+1}(K_{t+1},A)A}=M$, where M is a constant, can be rewritten as:

$$A^{\alpha}\frac{x}{[\theta_{t+1}(1/x)]^{1-\alpha}}=M.$$

After some algebra, one can check that the left hand side of the above equation increases in x. As a result, an increase in A reduces x and thus the relative size of the financial sector, while a drop in A does the reverse. Finally, consider the long run response. One can see from the Proof of Proposition 1 and from Equation (A1), financial intermediation drops in the long run and the relative size of the financial sector remains constant.

Consider now the effect of a change in trust Γ. The equilibrium condition is the same as the one represented above. Because the function $\theta_{t+1}(K_{t+1}, \Gamma)$ increases in Γ, higher trust increases investment and intermediation, while a drop in trust does the reverse. Accordingly, because also the function $\theta_{t+1}(1/x, \Gamma)$ increases in Γ, an increase in trust on impact increases the relative size of the financial sector while a reduction in trust does the reverse. Finally, in the Proof of Proposition 1 we also establish that long run intermediation and the long run relative size of finance increase in trust.

Proof of Lemma 2. We studied fee setting for $\Gamma \geq \Delta_t/2$. Consider the case $\Gamma < \Delta_t/2$. Now each manager monopolizes investment by all households located at distance less than or equal to Γ. Under a uniform distribution, each manager attracts a measure of 2Γ households, for a total of $m_t 2\Gamma = \Gamma/(\Delta_t/2)$. The remaining $1 - \Gamma/(\Delta_t/2)$ households do not participate in risk taking.

In this setting, the optimal fee set by each monopolistic manager maximizes:

$$2 \cdot w_{t-1} \cdot f_{jt} \cdot \int_0^\Gamma (\Gamma - \delta) \cdot \frac{\mathbb{E}\left(R_t - \gamma - f_{jt}\right)}{\sigma_t} \cdot d\delta,$$

which yields an optimal fee of $f_t^* = \frac{\mathbb{E}(R_t - \gamma)}{2} \equiv \varphi \cdot \mathbb{E}(R_t - \gamma)$ where $\varphi = 1/2$. The wealth invested by the households participating in risk taking is equal to:

$$\iint_{i,j} w_{t-1} \theta_{ij}(f_{jt})\, didj = w_{t-1} \cdot m_t \cdot 2 \cdot \left[(1-\varphi) \cdot \frac{\mathbb{E}(R_t - \gamma)}{\sigma_t} \cdot \int_0^\Gamma (\Gamma - \delta) d\delta\right] =$$

$$= w_{t-1} \cdot \frac{1}{\Delta_t} \frac{\mathbb{E}(R_t - \gamma)}{2\sigma_t} \cdot \frac{\Gamma^2}{2}.$$

By Equation (17), as the capital stock increases (i.e. K_t goes up), there is entry of money managers. This causes Δ_t to go down. As a result, the number of individuals participating in risk taking $\Gamma/(\Delta_t/2)$ also increases. Individuals who were already taking risk continue to do so, and invest larger absolute amounts owing to their higher wages. If the capital stock keeps increasing, and entry of new intermediaries continues, at some point $\Delta_t/2 < \Gamma$. From this point onward, the equilibrium fee is the corresponding one in Equation (16). The remaining comparative statics then follow by inspection of Equations (16) and (17).

Proof of Proposition 2 With endogenous entry, the evolution of the economy is described by the following equations:

$$K_t = \frac{(1 + \alpha A K_t^{\alpha-1} - \gamma)}{\sigma[1 + AK_t^{\alpha-1}]^2} \cdot (1 - \varphi_t) \cdot \left(\Gamma - \frac{\Delta_t}{4}\right) \cdot (1-\alpha) A K_{t-1}^{\alpha}, \tag{A5}$$

$$\Delta_t \cdot \varphi(\Delta_t) \cdot \left[\frac{(1-\gamma)}{A} K_t^{1-\alpha} + \alpha\right] = \eta, \tag{A6}$$

for $\Delta_t/2 > \Gamma$, and

$$K_t = \frac{(1+\alpha A K_t^{\alpha-1} - \gamma)}{\sigma[1 + A K_t^{\alpha-1}]^2} \cdot (1-\varphi_t) \cdot \left(\Gamma - \frac{\Delta_t}{4}\right) \cdot (1-\alpha) A K_{t-1}^{\alpha}, \qquad (A5')$$

$$\Delta_t \cdot \varphi(\Delta_t) \cdot \left[\frac{(1-\gamma)}{A} K_t^{1-\alpha} + \alpha\right] = \eta. \qquad (A6')$$

for $\Delta_t/2 < \Gamma$. Equations $(A5)$ and $(A5')$ are essentially the same law of motion of the Proof of Proposition 1, with the only difference that now Δ_t (and thus φ_t) are endogenously determined in Equations $(A6)$ and $(A6')$. In the spirit of the Proof of Proposition 1, we can rewrite (A5) as:

$$K_t^{\alpha} \frac{\sigma[K_t^{1-\alpha} + A]^2}{\left[\frac{(1-\gamma)}{A} K_t^{1-\alpha} + \alpha\right]} \cdot \frac{1}{(1-\varphi_t) \cdot \left(\Gamma - \frac{\Delta_t}{4}\right)} = (1-\alpha) A^2 K_{t-1}^{\alpha}. \qquad (A7)$$

Consider first the case where $\Delta_t/2 < \Gamma$. By replacing in Equation $(A6)$ the expression for $\varphi(\Delta_t)$ and by denoting $s(x) \equiv \left[\frac{(1-\gamma)}{A} x + \alpha\right]$, we can find after some algebra that

$$\left(\frac{\Delta_t}{\Gamma}\right)^2 - \frac{1}{4}\left(\frac{\Delta_t}{\Gamma}\right)^3 = \left[\frac{\eta}{\Gamma s(x)}\right],$$

where $x \equiv K_t^{1-\alpha}$. This equation has a unique solution for Δ_t/Γ in (0,1) which we denote by $\psi(x)$.

By replacing the expression for $\psi(x)$ in the expressions for φ_t and Δ_t in Equation $(P7)$, we find after some algebra that the law of motion of the economy is given by:

$$K_t^{\alpha} \cdot \frac{\sigma[x + A]^2}{\Gamma \cdot s(x)\left[1 - \psi(x) + \frac{\psi(x)^2}{4}\right] \cdot \left[1 - \frac{\psi(x)}{4}\right]} = (1-\alpha) A^2 K_{t-1}^{\alpha}, \qquad (A8)$$

Here again we have that $x \equiv K_t^{1-\alpha}$. The above difference equation has one trivial steady state at $K_t = x = 0$. A positive and unique steady state exists provided: i) the root multiplying K_t^{α} on the left hand side above is monotonically increasing in x, ii) the value of the root at $x = 0$ is below $(1-\alpha)A^2$. The latter condition is met when the variance σ is sufficiently low. On the other hand, a sufficient condition for i) is that:

$$s'(x) = \frac{(1-\gamma)}{A} \quad \text{is sufficiently small.}$$

Intuitively, in this case the main effect of higher x is to increase the numerator, leaving the denominator almost unaffected (also because in this case $\psi'(x)$ stays small). When this is the case, there is a unique interior equilibrium $K^* > 0$. This equilibrium is locally stable (so that the capital stock monotonically converges to it) provided the slope of the implicit mapping $K_t(K_{t-1})$ is above one at the $K^* = 0$ steady state. One can check that this is the case provided A is sufficiently high and σ is above a threshold (consistent with the previous upper bound). The condition that σ be bounded is the same as the one required in Proposition 1, except that now the bounds are evaluated at the equilibrium number of managers prevailing when $x = 0$ as entailed by $\psi(0)$. Since $\psi(0)$ does not depend on productivity A, the assumption that A be sufficiently large can be added to ensure stability of the system. Note that when $\psi'(0)$ is made small, the upper and lower bound will be consistent because locally entry responds slowly to changes in the capital stock, so that around $x = 0$ the analysis does not virtually change from that with a fixed number of money managers. It is immediate to see that the same condition is sufficient for stability when $\Delta_t/2 > \Gamma$. The intuition is that also in this case a variant of Equation (A7) holds, except that now the fee φ_t is fixed. Thus, the condition that $s'(x)$ be small is sufficient to guarantee that the $\psi'(x)$ holding under the fixed fee assumption is small as well. Here $\psi'(x)$ is smaller because changes in x leave the fee unchanged.

ONLINE APPENDIX B: EXTENSIONS

B.1 Technical Progress

We allow for productivity augmenting technological progress by assuming that the effective labor supply available at time t satifies the law of motion:

$$L_t = (1+n)(1+x)L_{t-1},$$

where n is the rate of population growth and x is the rate of technical progress. Because the production function is Cobb-Douglas, this formulation of labor augmenting technical progress is equivalent to one in which productivity growth is factor-neutral and increases the value of A.

Denoting by $\widehat{K}_t \equiv K_t/L_t$ the capital stock per unit of effective labor, the competitive remunerations of a unit of effective labor and of a unit of capital are respectively given by:

$$(1-\alpha)A\widehat{K}_t^{\alpha} = w_t,$$

$$\mathbb{E}\{R_t\} = 1 + \alpha A\widehat{K}_t^{\alpha-1},$$

and where the variance of the return to capital is equal to $\sigma_t = var(R_t) = \sigma\left[1 + A\widehat{K}_t^{\alpha-1}\right]^2$. The share of wage income invested into risky asset also depends on $\widehat{K}_t$, namely:

$$\theta_t = \frac{(1-\varphi)\left(1+\alpha A\widehat{K}_t^{\alpha-1} - \gamma\right)}{\sigma\left[1 + A\widehat{K}_t^{\alpha-1}\right]^2} \cdot \left(\Gamma - \frac{\Delta}{4}\right).$$

The total value K_t of the capital stock created at t is equal to $K_t = \theta_t \cdot w_{t-1} \cdot L_{t-1}$. Thus, the law of motion of the capital stock per unit of effective labor is given by:

$$\widehat{K}_t = \frac{\theta_t}{(1+n)(1+x)} \cdot (1-\alpha)A\widehat{K}_{t-1}^{\alpha}.$$

In light of the previous analysis, several immediate consequences follow. First, the capital stock per unit of effective labor converges to a nonzero steady state value $\widehat{K}^*$ that is a decreasing function of n and x. In this steady state, the per-capita capital stock and per capita output grow at a constant rate x, while the extent of risk taking θ_t converges to a constant. The comparative statics properties described by Proposition 1 continue to hold with respect to the steady state levels of the per capita capital stock and of the extent of risk taking. Second, the properties of evolution of the financial sector also do not change from Corollary 1. The management fee per unit of capital declines over time as $\widehat{K}_t$ increases toward its steady state level. As a consequence, financial sector income rises faster than value added if we express both the numerator and the denominator in per effective units of labor. Finally, the qualitative properties of Corollary 2 also hold in this modified model. In sum, population and productivity growth introduce additional reasons for the growth of the absolute size and profits of the financial sector, but do not affect the qualitative behavior of scaled variables such as unit fees and the income share going to finance.

B.2 Trading and Valuation of the Capital Stock

In our baseline model consumption and capital are the same good, so that the elderly consume the capital stock they own at the end of their lives. This assumption simplifies the analysis, but it raises the issue of whether our result are robust to the more realistic setting in which capital cannot be converted back into consumption and so the elderly must sell their capital stock to the young. To shed light on this issue, suppose now that the consumption can be transformed into capital but capital cannot be converted back into consumption. This implies that at time t the elderly of the generation born at time $t-1$ must sell the economy's capital stock to the current young generation. The amount of capital held by the elderly at the end of time t is equal to $\varepsilon_t \cdot K_t$. If the price of capital in terms of consumption is p_t, the value at time t of the supply of capital in terms of consumption goods is equal to $p_t \cdot \varepsilon_t \cdot K_t$. On the demand side, the consumption income available to the young born at time t to buy – through money managers – the entire capital stock from the elderly is equal to $\theta_{t+1} \cdot w_t$. Of course, the young only demand capital from the elderly if the price of existing capital is not higher than the resource cost of creating new capital, i.e. provided $p_t \leq 1$, which importantly affects equilibrium prices.

To find the equilibrium price p_t, we must determine whether the capital stock $\varepsilon_t \cdot K_t$ available at time t is below or above the desired investment $\theta_{t+1} \cdot w_t$ by the young born at t. If the young wish to increase the stock of capital, namely $\varepsilon_t \cdot K_t < \theta_{t+1} \cdot w_t$, the equilibrium price of capital settles at $p_t = 1$ so as to make savers indifferent between buying existing capital goods and creating new ones. If instead the young wish to reduce the stock of capital, namely $\varepsilon_t \cdot K_t > \theta_{t+1} \cdot w_t$, then the new capital goods will not be produced and the price drops to $p_t = \frac{\theta_{t+1} \cdot w_t}{\varepsilon_t \cdot K_t} < 1$ so as to equate the values of the demand and the supply of capital goods.

Because our main results focus on transitions occurring below the steady state, let us consider the implications of this analysis for changes in the valuation of capital markets during these transitions. Recall that in these transitions, the desired capital stock increases over time, namely $K_{t+1} = \theta_{t+1} \cdot w_t > K_t$. As a consequence, if the potential shocks ε_t are sufficiently small that below the steady state capital the condition $\varepsilon_t \cdot K_t < \theta_{t+1} \cdot w_t$ holds (at least when K_t is far enough from the steady state), then during the transitional growth phase the unit price of capital stays constant at $p_t = 1$. In each period, the elderly sell their capital $\varepsilon_t \cdot K_t$ to the young, who add extra investment to implement their desired capital stock $\theta_{t+1} \cdot w_t$. The ex-post shock ε_t affects consumption by the elderly and new investment by the young, but leaves the aggregate capital stock next period unaffected. The law of motion of the economy is then identical to Equation (11): the possibility to trade capital goods does not affect how the economy converges to the steady state.

The possibility of trading in capital goods, however, affects the interpretation of our results. In particular, the capital stock K_t can now be interpreted as the market valuation of the aggregate wealth of the economy. The fact that the income share of the financial sector raises with K_t can then be viewed as the product of increasing capital market valuations. It should be noted, however, that in our model these valuations rise through the extensive margin – as new investment takes place – and not through increases in their unitary valuation p_t, which remains constant at 1.

B.3: Competitive Entry of Intermediaries and the Growth of Financial Sector Income

We now show that it is possible that the unit cost of finance (the ratio of financial sector income over financial assets):

$$f_t^* \theta_t = \varphi_t(\Delta_t) \cdot (1 - \varphi_t(\Delta_t)) \cdot \left(\Gamma - \frac{\Delta_t}{4}\right) \cdot \frac{(1 + \alpha A K_t^{\alpha-1} - \gamma)^2}{\sigma[1 + A K_t^{\alpha-1}]^2},$$

may increase over time, as new intermediaries enter the market. To see why this may be the case, note that during transitional growth, the capital stock K_t increases while the distance between managers Δ_t decreases. As a result, a sufficient condition for the product $f_t^* \theta_t$ to increase over time is that the terms that are functions of Δ_t decrease in Δ_t while ratio which is a function of K_t increases in K_t. It is immediate to see that the ratio on the right increases in K_t provided $\alpha < 1 - \gamma$. On the other hand, one can find values such that the first term (which is a polynomial of degree 5) decreases in Δ_t (e.g. Δ_t close to Γ). It is beyond the scope of this analysis to evaluate under what exact conditions unit costs may be increasing, but it seems that – given that Δ_t is pinned down by η – one may be able to find economies (values of η and of the initial capital stock) for which the equilibrium Δ_t is indeed close to Γ and unit costs increase over time until the steady state is reached.

ONLINE APPENDIX C: BUBBLES

Suppose now that newborns can take financial risk not only by investing in the economy's capital stock, but also in a non-fundamental "bubbly" asset. It is easiest to think of this assets as just a risky pyramid scheme. A newborn buying one unit of this asset at t is entitled to receive a payment next period equal to his pro-rata share of the total market value of the same asset at $t + 1$. The future value of the bubble is uncertain at t because of volatility in agents' beliefs about the bubble's future value. Similarly to physical capital, then, the bubble is a risky investment that requires delegation to a trusted intermediary.

Suppose that the aggregate value of the bubble bought by newborns at t is equal to B_t. Then each newborn at t spends on the bubble an amount equal to $b_t = B_t/L_t$. If at $t + 1$ the aggregate value of the bubble is $b_{t+1}L_{t+1}$, each of the now elderly receives from the L_{t+1} newborns an amount of consumption equal to $b_{t+1}(L_{t+1}/L_t) = b_{t+1}(1 + n)$. The return from purchasing the bubble for an agent born at time t is thus equal to $(b_{t+1}/b_t)(1 + n)$. As of time t, the expected gross return from investing in the bubble is then equal to:

$$\frac{\mathbb{E}(b_{t+1})}{b_t}(1 + n).$$

The investor's net return subtracts from the above expression the management fee.

The expectation $\mathbb{E}(b_{t+1})$ depends on the process governing agents' beliefs. This process also pins down the risk entailed in the bubbly investment. For simplicity and to illustrate the basic idea, we assume that, at any t, newborns believe that the future value of the bubble is perfectly positively correlated with the future productivity of capital and that the variance of the return on the bubble equals the variance of the return to capital. This assumption captures the idea that the bubble effectively reflects an overvaluation of some firms in the economy, so that it co-moves with the fundamental value of capital. This formulation greatly simplifies the analysis because it implies imply that the bubble and the capital stock are perfect substitutes for the purpose of risk taking.

In particular, in equilibrium the expected return on the bubble is equalized to that on physical capital, managers charge the same fee on the two assets, and newborns select how much overall risk to take. The portfolio shares on the bubbly asset and on the capital stock are then endogenously determined by the market value of these assets. In this case, the laws of motion of the capital stock per effective unit of labor and of the bubble satisfy the following equations:

$$\frac{\mathbb{E}(b_{t+1})}{b_t}(1+n) = 1 + \alpha \cdot A \cdot \widehat{K}_t^{\alpha-1}, \tag{C1}$$

$$\widehat{K}_{t+1}(1+n) = \theta_{t+1} \cdot (1-\alpha) A \widehat{K}_t^{\alpha} - b_t. \tag{C2}$$

Equation (C1) states that the expected return on the bubble is equal to the expected return on capital; Equation (C2) shows how the bubble crowds out some real investment.

To illustrate the impact of the bubble on finance income, we focus on the steady state $(b^*, \widehat{K}^*)$. The steady state is described by an expected value b^* around which the per worker bubble fluctuates, and an expected value $\widehat{K}^*$ around which capital per worker fluctuates. These values are pinned down by the system of equations:

$$\alpha \cdot A \cdot \left(\widehat{K}^*\right)^{\alpha-1} = n,$$

$$b^* = \theta^* \cdot (1-\alpha) \cdot A \cdot \left(\widehat{K}^*\right)^{\alpha} - \widehat{K}^*(1+n),$$

subject to the condition $b^* > 0$, which is necessary for the existence of positive bubbles.

Proposition 3 *There exist two thresholds $\underline{n}$ and $\overline{n}$, where $\underline{n} < \overline{n}$, such that for $n \in (\underline{n}, \overline{n})$ there exists a bubbly steady state $(b^*, \widehat{K}^*)$ with $b^* > 0$, in which:*

i) *The capital stock is smaller and the return to capital is higher than in the bubble-less equilibrium of Section 5.2.*

ii) *The finance income share $\varphi \cdot (1+n-\gamma) \cdot \frac{(\widehat{K}^*+b^*)}{A \cdot (\widehat{K}^*)^{\alpha}}$ is larger than in the bubble-less equilibrium of Section 5.2.*

Proof: A bubble-less equilibrium is identified by a per capita capital stock level $\widehat{K}_{nb}$ satisfying:

$$\widehat{K}_{nb}(1+n) = \theta_{nb} \cdot (1-\alpha) \cdot A \widehat{K}_{nb}^{\alpha}.$$

A bubbly equilibrium is identified by a vector $(b^*, \widehat{K}^*)$ satisfying the system of equations:

$$\alpha \cdot A \cdot \left(\widehat{K}^*\right)^{\alpha-1} = n,$$

$$b^* = \theta^* \cdot (1-\alpha) \cdot A \cdot \left(\widehat{K}^*\right)^{\alpha} - \widehat{K}^*(1+n),$$

subject to the condition $b^* > 0$. By plugging the equilibrium condition $\widehat{K}^* = (\alpha \cdot A/n)^{1/(1-\alpha)}$ in the equation for b^* we find that the equilibrium admits a positive bubble if and only if:

$$\left(\frac{1-\varphi}{\sigma}\right)\left(\Gamma - \frac{\Delta}{4}\right)\alpha(1-\alpha) > \frac{1+n}{n}\frac{(\alpha+n)^2}{(1+n-\gamma)}.$$

After some algebra, one can check that under the condition $2\alpha > (1-\gamma)$, the left hand side of the above expression is U-shaped in n. But then, since the left hand side diverges both for $n \to 0$ and for $n \to \infty$, there are two thresholds n_* and n^*, where $n_* < n^*$, such that a bubbly equilibrium exists if and only if $n \in (n_*, n^*)$. Note that when $n > n_*$ the economy is dynamically inefficient, in the sense that $\alpha A \widehat{K}_{nb}^{\alpha-1} < n$.

Finance income is higher in the bubbly than in the bubble-less equilibrium if and only if:

$$\varphi \cdot (1 + n - \gamma) \cdot \frac{\widehat{K}^* + b^*}{A \cdot \left(\widehat{K}^*\right)^{\alpha}} > \varphi \cdot \left(1 + \alpha A \widehat{K}_{nb}^{\alpha-1} - \gamma\right) \cdot \frac{\widehat{K}_{nb}}{A \widehat{K}_{nb}^{\alpha}}.$$

Given that when the bubble exists we have that $\alpha A \widehat{K}_{nb}^{\alpha-1} < n$, a sufficient condition for the bubble to expand financial income is that:

$$\frac{\widehat{K}^* + b^*}{A \cdot \left(\widehat{K}^*\right)^{\alpha}} > \frac{\widehat{K}_{nb}}{A \widehat{K}_{nb}^{\alpha}} \Leftrightarrow \theta^* \cdot (1 - \alpha) - \frac{n}{A} \left(\widehat{K}^*\right)^{1-\alpha} > \theta_{nb} \cdot (1 - \alpha) - \frac{n}{A} \widehat{K}_{nb}^{1-\alpha}.$$

Given that $\widehat{K}^* < \widehat{K}_{nb}$, a sufficient condition for the above inequality is that the bubble encourages risk taking, namely that $\theta^* > \theta_{nb}$. It is easy to see that this condition holds provided the increase in expected returns caused by the bubble more than offsets the increases risk σ_t (where the latter effect occurs because the marginal product of capital, and thus its fluctuations, increase with the bubble). A sufficient condition for $\theta^* > \theta_{nb}$ to hold is that risk taking:

$$\theta = (1 - \varphi) \cdot \left(\Gamma - \frac{\Delta}{4}\right) \alpha^2 \frac{(1 + y - \gamma)}{\sigma[\alpha + y]^2}$$

Increases with the marginal product of capital in value added $y = \alpha \cdot A \cdot K^{\alpha-1}$. This is indeed the case provided $\alpha \cdot A \cdot K^{\alpha-1} < \alpha - 2(1 - \gamma)$. But then, given that the highest marginal return of capital is attained at the bubbly steady state, a sufficient condition for $\theta^* > \theta_{nb}$ to hold is that $n < n^{**} \equiv \alpha - 2(1 - \gamma)$. It is easy to see that $n^{**} > n_*$. By defining $\underline{n} \equiv n_*$ and $\overline{n} \equiv \min(n^{**}, n^*)$, we can see that for $n \in (\underline{n}, \overline{n})$ the properties of Proposition 3 are verified.

As in the Samuelson and Tirole models, the bubble crowds out productive capital and raises the rate of return delivered by all financial assets. The bubble exists only if the economy is dynamically inefficient, which is guaranteed by the condition $n > \underline{n}$.[1] Population growth cannot however be too large (i.e. $n < \overline{n}$), for otherwise the returns of the capital stock and of the bubble would be too volatile, and individuals would be unwilling to hold the bubble.

The bubble expands the finance income share relative to the equilibrium without bubbles of Section 5.2, for two reasons. First, the bubble raises rates of return paid by all risky financial assets. This effect increases the unit fee that money managers can charge to their clients, and thus the total income earned by financial intermediaries. Second, the risky bubble constitutes an intermediated investment that crowds out productive capital. This effect reduces per capita income below the no-bubble equilibrium level, increasing the wealth income ratio and the finance share in income.

---

[1] Formally, this occurs when in the bubble-less equilibrium of Section 5.2 the steady state return to capital is below the population growth rate, namely $A \cdot \widehat{K}^{\alpha-1} < n$.