Prices and Returns in LAPM

Under the dividend growth model, price at time t of a stock of firm i with dividend $D_{i,t}$ and growth $g_{i,t}$ equals

$$P_{i,t}^{\theta} = \sum_{s\geq 1} \frac{\mathbb{E}_t^{\theta} D_{i,t+s}}{(1+\rho)^s}$$

where $D_{i,t+s} = D_{i,t} e^{\sum_{k=1}^{s} f_{i,t+k}}$. We can then show:

Proposition. *For* $a < 1$, and *provided the discount factor* $1+\rho$ *is sufficiently large, price is equal to:*

$$P_{i,t}^{\theta} = D_{i,t} \sum_{s\geq 1} \exp\left\{-A_0 s + \mathbb{E}_t^{\theta} f_{i,t} \frac{a(1-a^s)}{1-a} + A_2 \left(1-a^{2s}\right)\right\} \quad (1)$$

where A_0 *and* A_2 *are constants.*

Due to the presence of the exponential terms a^s and a^{2s}, this sum does does not have a closed form expression for $a > 0$. However, we can describe its properties under the approximation where the exponential terms are neglected, as below:

$$P_{i,t}^{\theta} = D_{i,t} \sum_{s\geq 1} \exp\left\{-A_0 s + \mathbb{E}_t^{\theta} f_{i,t} \frac{a}{1-a} + A_2 \right\} \quad (2)$$

The approximation is asymptotically valid for large s (since these terms decrease exponentially in s), but generates some error for the first few terms. However, this error tends to zero with a small. We therefore refer to (2) as the small-a approximation of (1).

Lemma. *In the approximation (2) we have:*

$$P_{i,t}^{\theta} = P_{i,t} \cdot e^{\frac{a\theta K}{1-a}(g_t - a\mathbb{E}_{t-1} f_{t-1})}$$

where $P_{i,t} = D_{i,t} \frac{e^{\mathbb{E}_t f_{i,t} \frac{a}{1-a} + A_2}}{(1+\rho)e^{-\frac{\sigma_\eta^2/2}{1-a^2}} - 1}$ *is the rational price.*

Besides providing a closed form solution, the small a approximation makes explicit the compounding nature of the error in expectation formation: the surprise today pollutes growth expectations far into the future, generating an exponentially discounted distortion to the rational price. Thus, $P_{i,t}^{\theta}$ deviates from the rational benchmark when three conditions are met: the process for fundamentals is persistent, $a > 0$, informative $K > 0$, and investors' beliefs are distorted by representativeness, $\theta > 0$. In this generic case, $P_{i,t}^{\theta}$ is too high after good news, $g_t > a\mathbb{E}_{t-1}f_{t-1}$, and is too low after bad news.

The Lemma has several implications. First, prices are excessively volatile relative to the dividend stream, as in Shiller (1984).

Corollary 1. *When $a\theta K > 0$, prices are excessively volatile:*[1]

$$\frac{var\left(P_{i,t}^{\theta}\right)}{var\left(P_{i,t}\right)} > 1$$

Second, returns are characterized as:

Corollary 2. *Returns are equal to:*

$$\frac{P_{i,t+1}^{\theta}}{P_{i,t}^{\theta}} = \frac{P_{i,t+1}}{P_{i,t}} e^{\frac{a\theta K}{1-a}[(g_{t+1}-g_t)-a(\mathbb{E}_t f_t-\mathbb{E}_{t-1}f_{t-1})]}$$

The functional form is remarkably transparent. With diagnostic expectations, firms earn abnormal returns: they are too high (low) after increasing (decreasing) growth surprises. In particular, the HLTG portfolio has negative excess returns, while the LLTG portfolio has positive excess returns. Moreover, it follows that future returns are predictable.

[1] It would be nice to show $P_{i,t}^{\theta}$ can be more volatile than the realizations of $D_{i,t}$ itself.

Corollary 3. *Abnormal returns at t predictably reverse at $t+1$:*

$$\mathbb{E}_t \frac{P_{i,t+1}^{\theta}}{P_{i,t}^{\theta}} = e^{-\frac{a\theta K}{1-a}(g_t - a\mathbb{E}_{t-1}f_{t-1})} e^{V(\sigma_f^2, \sigma_\varepsilon^2, \theta)} \qquad (3)$$

Two features of expression (3) are noteworthy. The first exponential shows that expected returns at $t+1$ undo the excess growth of prices at t. Second, in the absence of this reversal, returns are positive in the rational benchmark because shocks to growth are log-normal and have positive expectation. However, diagnostic expectations boost this effect.

Proofs.

Proposition.

Because the $f_{i,t+k}$ are jointly Gaussian, the random variables $F_{i,t,t+s} \equiv \sum_{k=1}^{s} f_{i,t+k}$, defined for any finite s, are also normal. Denote the respective mean and variance by $\bar{F}_{i,t,t+s}$ and $\sigma^2_{F_{i,t,t+s}}$. We then have:

$$P_{i,t} = D_{i,t} \sum_{s\geq 1} \frac{\mathbb{E}_t e^{F_{i,t,t+s}}}{(1+\rho)^s} = D_{i,t} \sum_{s\geq 1} \frac{e^{\bar{F}_{i,t,t+s} + \frac{1}{2}\sigma^2_{F_{i,t,t+s}}}}{(1+\rho)^s}$$

provided the sum converges. It is straightforward to show that

$$\bar{F}_{i,t,t+s} = f_{i,t} \sum_{k=1}^{s} a^k = f_{i,t} a \frac{1-a^s}{1-a}$$

To compute the variance, we first note that:

$$f_{i,t+k} = \sum_{l=1}^{k} a^k f_{i,t} + a^{k-l}\eta_{t+l}$$

We then write $\sigma^2_{F_{i,t,t+s}}$ as:

$$var\left(\sum_{k=1}^{s} f_{i,t+k}\right) = var\left(\sum_{k=1}^{s}\sum_{l=1}^{k} a^{k-l}\eta_{t+l}\right)$$

Regrouping terms, we find:

$$var\left(\sum_{l=1}^{s}\eta_{t+l}\sum_{k=l+1}^{s} a^{k-l}\right) = \sigma_\eta^2\sum_{l=1}^{s}\sum_{k=l+1}^{s} a^{2k-2l}$$

because $cov(\eta_{t+l},\eta_{t+l'}) = \delta_{l,l'}\sigma_\eta^2$. We then have:

$$\sigma_{F_{i,t,t+s}}^2 = \frac{\sigma_\eta^2}{1-a^2}\left(s + a^2\frac{1-a^{2s}}{1-a^2}\right)$$

As expected, this converges to $s\sigma_\eta^2$ when $a = 0$, and increases in a. Inserting into the price expression above, we find:

$$P_{i,t} = D_{i,t}\sum_{s\geq 1}\exp\left\{f_{i,t}a\frac{1-a^s}{1-a} + \frac{1}{2}\frac{\sigma_\eta^2}{1-a^2}\left(s + a^2\frac{1-a^{2s}}{1-a^2}\right) - s\ln(1+\rho)\right\}$$

$$= D_{i,t}\sum_{s\geq 1}\exp\left\{s\left(\frac{1}{2}\frac{\sigma_\eta^2}{1-a^2} - \ln(1+\rho)\right) + A_1(1-a^s)\right.$$

$$\left. + A_2\,(1-a^{2s})\right\}$$

where

$$A_1 = \frac{f_{i,t}a}{1-a}, \quad A_2 = \frac{1}{2}\frac{\sigma_\eta^2 a^2}{(1-a^2)^2}$$

For $a < 1$, this expression converges (absolutely) provided discounting is sufficiently large relative to the conditional variance,

$$\ln(1+\rho) > \frac{1}{2}\frac{\sigma_\eta^2}{1-a^2}$$

However, the sum above does not have a closed form expression for $a > 0$. This is due to the presence of the exponential terms a^s and a^{2s}. However, we can describe its properties under approximation (2) described in the text.

Lemma. Note that as s increases the terms a^s and a^{2s} decrease exponentially (while the term in s increases linearly). Neglecting these terms, we write

$$P_{i,t} = D_{i,t} e^{A_1 + A_2} \sum_{s \geq 1} \exp\left\{ s \left(\frac{\sigma_\eta^2 / 2}{1 - a^2} - \ln(1+\rho) \right) \right\} = \frac{D_{i,t} e^{A_1 + A_2}}{(1+\rho) \mathrm{e}^{-\frac{\sigma_\eta^2/2}{1-a^2}} - 1}$$

Making explicit the role of diagnostic expectations, we write:

$$P_{i,t}^{\theta} = \frac{D_{i,t} e^{\frac{f_{i,t} a}{1-a} + \frac{1}{2} \frac{\sigma_\eta^2 a^2}{(1-a^2)^2}}}{(1+\rho) \mathrm{e}^{-\frac{\sigma_\eta^2/2}{1-a^2}} - 1} e^{\frac{a \theta K}{1-a}(g_t - a \mathbb{E}_{t-1} f_{t-1})}$$

The advantage of this approximation is that it makes explicit the compounding nature of the error in expectation formation: the surprise today pollutes growth expectations far into the future, and its effect on price is summarized by the exponential $e^{\theta K (g_t - a \mathbb{E}_{t-1} f_{t-1}) \frac{a}{1-a}}$.

Another possible approximation is to expand the summand for small θ. Note that only the term A_1 depends on $f_{i,t}$ and is thus modified under diagnostic expectations. Using $f_{i,t}^{\theta} = f_{i,t} + \theta K (g_t - a \mathbb{E}_{t-1} f_{t-1})$, we find:

$$P_{i,t}^{\theta} = \sum_{s \geq 1} \frac{\mathbb{E}_t D_{i,t+s}}{(1+\rho)^s} \cdot \left(1 + \theta K (g_t - a \mathbb{E}_{t-1} f_{t-1}) \frac{a(1-a^s)}{1-a} \right)$$

$$= P_{i,t} + a \theta K \Lambda_t (g_t - a \mathbb{E}_{t-1} f_{t-1})$$

where $\Lambda_t = \sum_{s \geq 1} \frac{\mathbb{E}_t D_{i,t+s}}{(1+\rho)^s} \cdot \frac{1-a^s}{1-a} > 0$. However, this approximation does not yield closed form expressions.

Corollary 1. [to complete]

Corollary 2. In the small a approximation, we find:

$$\frac{P_{i,t+1}^{\theta}}{P_{i,t}^{\theta}} = \frac{P_{i,t+1}}{P_{i,t}} e^{\frac{a\theta K}{1-a}[(g_{t+1}-g_t)-a(\mathbb{E}_t f_t-\mathbb{E}_{t-1}f_{t-1})]}$$

In the approximation of small θ we find:

$$\frac{P_{i,t+1}^{\theta}}{P_{i,t}^{\theta}} \sim \frac{P_{i,t+1}}{P_{i,t}} + a\theta K\Lambda_t[(g_{t+1}-g_t)-a(\mathbb{E}_t f_t-\mathbb{E}_{t-1}f_{t-1})]$$

Corollary 3. Denote $\Lambda = (1+\rho)\mathrm{e}^{-\frac{\sigma_\eta^2/2}{1-a^2}} - 1$ and $A = ak(1+\theta)$. We then have:

$$\frac{\mathbb{E}_t P_{i,t+1}^{\theta}}{P_{i,t}^{\theta}} = \frac{1}{P_{i,t}} \frac{D_t e^{\mathbb{E}_t f_t \frac{a(a-A)}{1-a}+A_2}}{\Lambda} \mathbb{E}_t\left(e^{f_{t+1}+\frac{A}{1-a}g_{t+1}}\right) =$$

$$= e^{-a\mathbb{E}_t f_t\left(1+\frac{A}{1-a}\right)} e^{\mathbb{E}_t f_{t+1}\left(1+\frac{A}{1-a}\right)} \cdot e^{-\frac{a\theta K}{1-a}[g_t-a\mathbb{E}_{t-1}f_{t-1}]}$$

$$\cdot e^{\frac{\left(1+\frac{A}{1-a}\right)^2}{2}\sigma_f^2+\frac{\left(\frac{A}{1-a}\right)^2}{2}\sigma_\varepsilon^2} = e^{-\frac{a\theta K}{1-a}[g_t-a\mathbb{E}_{t-1}f_{t-1}]} \cdot e^{V\left(\sigma_f^2,\sigma_\varepsilon^2,\theta\right)}$$

where we defined $V\left(\sigma_f^2,\sigma_\varepsilon^2,\theta\right) = \frac{\left(1+\frac{A}{1-a}\right)^2}{2}\sigma_f^2 + \frac{\left(\frac{A}{1-a}\right)^2}{2}\sigma_\varepsilon^2$.

In the small θ approximation, we have instead

$$\mathbb{E}_t \frac{P_{i,t+1}^{\theta}}{P_{i,t}^{\theta}} \sim \mathbb{E}_t \frac{P_{i,t+1}}{P_{i,t}} - \theta \cdot aK\Lambda_t[g_t - a\mathbb{E}_{t-1}f_{t-1}]$$

After a positive surprise at time t, returns in $t+1$ are too low. The converse happens after a negative surprise.